Weil divisors, Class group

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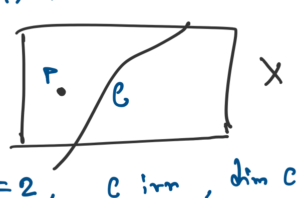
Recall: for an irreducible closed subscheme $Y \subseteq X$,

$\operatorname{codim}(Y, X) = \sup\{ n \mid$ There is a strict chain of irr closed subsets of X
$Y = Z_0 \subsetneq Z_1 \subsetneq \cdots \subseteq Z_n \subseteq X \}$

For any closed subset $Y_1 \subseteq X$.
$\operatorname{codim}(Y_1, X) = \inf\{ \operatorname{codim}(Z, X) \mid Z \subseteq Y_1 \text{ closed}\}$.

eg:



X irr, $\dim X = 2$, C irr, $\dim C = 1$. Set $Y = \{p\} \cup C$ where p is a point not in C.
Then $\operatorname{codim}(Y, X) = 1$.

Fact: R noetherian domain, $f \in R$ non-zero. Then each irr component of $V(f)$ has codim 1.
This follows from Krull's height Theorem. (exercise).

Exercise: 1) Let ξ be the generic pt of an irr closed subset $Z \subseteq X$.
Then $\operatorname{codim}(Z, X) = \dim \mathcal{O}_{X,\xi}$.

2) For $p \in X$,
$\operatorname{codim}(\overline{\{p\}}, X) = \dim \mathcal{O}_{X,p}$.

Def. For $p \in X$, define codimension of X at p to be $\operatorname{codim}(\overline{\{p\}}, X) = \dim \mathcal{O}_{X,p}$

Def: X be an integral scheme.
1) A prime divisor on X is an integral closed subscheme of X of codimension 1.

2) A Weil divisor on X is a finite formal $\mathbb{Z}$-linear combination of prime divisors on X.

3) Denote the free abelian group generated by all prime divisors by $\operatorname{Div}(X)$.
So Weil divisors are members of $\operatorname{Div}(X)$.

E.g Consider $D = V(xy) - 2V(z) \subseteq \mathbb{A}^3 = \operatorname{Spec}(\mathbb{C}[x,y,z])$

$D = V(x) + V(y) - 2V(z)$ is a Weil divisor.

Eg: $\operatorname{Div}(\mathbb{A}^1_{\mathbb{C}}) =$ free abelian group generated by $\mathbb{C}$.

Def. A scheme X is called regular in codim 1 if for every $x \in X$ such that the local ring $\mathcal{O}_{X,x}$ has dim 1, $\mathcal{O}_{X,x}$ is regular.

We say X satisfies $(*)$ if X is a noetherian integral scheme which is regular in codim 1.

Fact. Normal integral schemes are regular in codimension 1.

Divisor of a rational function.

Recall: Let A be a regular local ring of dimension 1.
A is a domain,
The maximal ideal of A, $m_A = (t)$ for some $t \in A$.

Facts. 1) for $0 \neq a \in A$, $\exists!\, m \in \mathbb{Z}_{\geq 0}$ such that
$a = t^m \cdot u$ for some unit in A.

2) Thus for $0 \neq f \in \operatorname{Frac}(A)$, $\exists!\, m_0 \in \mathbb{Z}$ such that
$f = t^{m_0} \cdot u$ where u is a unit in A.

Def: Set $v(f) := m_0$.
This gives a map $v : \operatorname{Frac}(A) - \{0\} \longrightarrow \mathbb{Z}$
such that
$v(fg) = v(f) + v(g)$,
Set $v(0) := \infty$

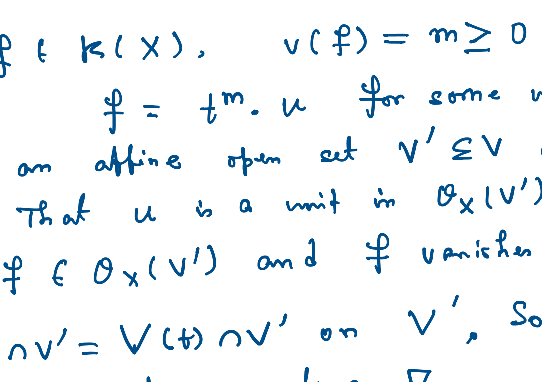
Fact. 1) v is independent of the choice of the generator t of m_A. (Exercise).
2) **Exercise:** Verify $m_A = \{ f \in \operatorname{Frac}(A) \mid v(f) > 0 \}$.

Discussion: Assume X satisfies condition $*$.
Z is a prime divisor on X; ξ be the generic point of Z.

Fact: $\dim(\mathcal{O}_{X,\xi}) = 1$. (Exercise).

So $\mathcal{O}_{X,\xi}$ is a regular ring of dimension 1.
So we have

$v_Z : \operatorname{Frac}(\mathcal{O}_{X,\xi}) = K(X) \longrightarrow \mathbb{Z}$



Suppose $t \in \mathcal{O}_{X,\xi}$ generates the maximal ideal of $\mathcal{O}_{X,\xi}$. Pick an affine nbhd V of ξ such that $t \in \mathcal{O}_X(V)$.

For $f \in K(X)$, $v(f) = m \geq 0$, in $\mathcal{O}_{X,\xi}$
$f = t^m \cdot u$ for some unit u in $\mathcal{O}_{X,\xi}$.
Pick an affine open set $V' \subseteq V$ containing ξ such that u is a unit in $\mathcal{O}_X(V')$.
So $f \in \mathcal{O}_X(V')$ and f vanishes along
$Z \cap V' = V(t) \cap V'$ on V', So f has a zero of order m along Z.

Similarly if $v(f) = -m$ for $m \geq 0$.
Then $v(1/f) = m$.
Thus f has a pole of order m along Z.

Ex. Take $X = \mathbb{A}^1_{\mathbb{C}}$, check X satisfies $*$.
For $x \in X$, understand the above discussion.

Lemma. Suppose X satisfies $*$. For nonzero $f \in K(X)$, $\{ Z \mid \operatorname{codim}(Z, X) = 1,\ v_Z(f) \neq 0\}$ is a finite set.
Recall $v_Z : K(X) \longrightarrow \mathbb{Z}$ is the valuation defined in the discussion above.

Pf. X is noetherian, so we can take a covering of X by finitely many affine opens in X:

$X = \bigcup_{i=1}^{n} U_i$

Fix $1 \leq i \leq n$, Say $U_i = \operatorname{Spec}(A_i)$.
$f \in \operatorname{Frac}(A_i)$. Choose $\alpha, \beta \in A_i$ such that
$f = \alpha/\beta$

Let $Z \subseteq X$ closed s.t $\operatorname{codim}(Z, X) = 1$ and $U_i \cap Z \neq \emptyset$; The closed subset $U_i \cap Z$ corresponds to a prime ideal P_Z of height one in A_i

Now $v_Z(f) = v_Z(\alpha) - v_Z(\beta) \neq 0$

$\Rightarrow$ either $v_Z(\alpha) >$ or $v_Z(\beta) > 0$

$\Rightarrow$ $\frac{\alpha}{1}$ or $\beta/1$ is in the maximal ideal of the local ring $(A_i)_{P_Z}$. The maximal ideal is $P_Z (A_i)_{P_Z}$.

$\Rightarrow$ $\alpha \in P_Z$ or $\beta \in P_Z$ in A_i. (†)

Now Krull's height Thm $\Rightarrow$ for $\gamma \in A_i$, the height one primes containing γ are the minimal primes containing γ. There are only finitely many such.

Thus from (†), we conclude that there are only finitely many Z with $\operatorname{codim}(Z, X) = 1$ and $Z \cap U_i \neq \emptyset$ such that $v_Z(f) \neq 0$.

Note for any codim 1 closed subset Z of X.
There exists some $1 \leq j \leq n$ s.t $Z \cap U_j \neq \emptyset$.
Then $v_Z = v_{Z \cap U_i}$ on $K(X) = k(U_i)$.
[Verify].

The next definition makes sense thanks to the lemma above.

Def. Suppose X satisfies $*$. For nonzero $f \in K(X)$, the divisor of f is defined to be

$$\operatorname{div}(f) := \sum_{\substack{Z \subseteq X \text{ closed} \\ \operatorname{codim}(Z,X) = 1}} v_Z(f) \cdot Z \in \operatorname{Div}(X)$$

Def Divisor of a rational function is called a principal divisor.

Rmk $\operatorname{div}(f)$ tracks the codimension one subsets along which f has a pole or zero along with the order of the pole or zero.

Exercise. X normal, integral, $f \in K(X)$.
If $v_Z(f) \geq 0$ for every codim 1 closed subset Z; Then $f \in \mathcal{O}(X)$.

Since $\operatorname{div}(fg) = \operatorname{div}(f) + \operatorname{div}(g)$, $\{\operatorname{div}(f) \mid f \neq 0, f \in K(X)\}$ is a subgp of $\operatorname{Div}(X)$.

Def: X satisfies $*$. The divisor class gp of X is defined to be

$$\operatorname{Cl}(X) := \frac{\operatorname{Div}(X)}{\{\operatorname{div}(f) \mid f \neq 0, f \in K(X)\}}$$

Eg: $X = \mathbb{A}^n_k = \operatorname{Spec}(k[x_1, \ldots, x_n])$

codim 1 closed subset of X $\xleftrightarrow[\text{one-to-one}]{}$ height one primes of $k[x_1, \ldots, x_n]$

$p \in \operatorname{Spec}(k[x_1, \ldots, x_n])$, $\operatorname{ht}(p) = 1$. choose non-zero $f \in p$ irreducible. Then (f) is prime $\Rightarrow p = (f)$

Thus the divisor $\sum_{i=1}^{n} m_i V(p_i) = \operatorname{div}\left(\prod_{i=1}^{n} f_i^{m_i}\right)$

So $\operatorname{Cl}(\mathbb{A}^n_k) = 0$ as every divisor is principal.

Fact: (prop 6.2 Hart).
A noetherian domain. $\operatorname{Cl}(\operatorname{Spec} A) = 0$ iff A is normal and a unique factorization domain.

Eg: $\operatorname{Cl}(\mathbb{A}^1_k - \{0\}) = \operatorname{Cl}(\operatorname{Spec} k[x, 1/x]) = 0$.

Eg. $X = \mathbb{P}^n_k$. $Z \subseteq X$ irr codim 1;

Then there is an irr homogeneous polynomial H_Z unique upto multiplication by an element of $k^\times$ s.t
$Z = V(H_Z) \subseteq \operatorname{Proj}(k[x_0, \ldots, x_n])$

Define $\deg(Z) = \deg H_Z$; extend to $\deg : \operatorname{Div}(X) \longrightarrow \mathbb{Z}$.

Recall: $K(X) = \{ a/b \mid a, b$ homo in $k[x_0, \ldots, x_n]$ of the same deg $\}$.

$\deg(D) = 0 \iff D = \sum m_i V(H_i) - \sum n_j V(G_j)$ such that each H_i, G_j are irr and, $m_i, n_j \geq 0$
$\sum m_i \deg H_i = \sum n_j \deg G_j$

Then $\lambda = \dfrac{\prod H_i^{m_i}}{\prod G_j^{n_j}} \in K(X)$

and $D = \operatorname{div}(\lambda)$

So

$$\begin{tikzcd} \operatorname{Div}(\mathbb{P}^n_k) \arrow[r, "\deg"] \arrow[d] & \mathbb{Z} \\ \operatorname{Cl}(\mathbb{P}^n) \arrow[ur, "\sim"'] & \end{tikzcd}$$

$$\boxed{\operatorname{Cl}(\mathbb{P}^n) = \mathbb{Z}}$$